TO THE NATURE OF IRREVERSIBILITY IN LINEAR SYSTEMS

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О ПРИРОДЕ НЕОБРАТИМОСТИ В ЛИНЕЙНЫХ СИСТЕМАХ

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Magnetic Resonance in Solids

Electronic Journal

Volume **6**, *No.* **1**,
pages **165-179**, **2004**

http://mrsej.ksu.ru

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New scenario of irreversibility for linear systems has been found and discussed. This scenario is based on the interpretation of the geometrical/physical meaning of the temporal fractional integral with complex and real fractional exponents. It has been shown that imaginary part of the fractional integral related to *discrete-scale invariance* (DSI) phenomenon and observed only for *true* regular (discrete) fractals. Numerical experiments show that the imaginary part of the complex fractional exponent can be well approximated by simple and finite combination of the leading sine/cosine log-periodical functions with period $\ln\xi$ (ξ is a scaling parameter). In the most cases analyzed the leading Fourier components give a pair of complex conjugated exponents defining the imaginary part of the complex fractional integral. For random fractals, where invariant scaling properties are realized only in the statistical sense the imaginary part of the complex exponent is *averaged* and the result is expressed in the form of the conventional Riemann-Liouville integral. The conditions for realization of reind and recaps elements with complex power-law exponents have been found. The fractal structures leading to pure log-periodic oscillations related to fractional integration with complex exponent are analyzed. Description of relaxation processes by kinetic equations containing complex fractional exponent and their possible recognition in the dielectric spectroscopy is discussed.

1. Introduction

As it is known [1], all equations describing the kinetic stage of evolution should have clearly expressed irreversibility with respect to the temporal variable *t*. This inherent irreversible property should be derived from the basic reversible equation for density matrix (Liouville equation). The correct procedure of such deduction is reduced to some decoupling procedure as it is done usually in the Mori-Zwanzig formalism [2] or to the procedure of the reduced description based on the hypothesis of an intermediate local equilibrium. The last hypothesis is the basic one in the Zubarev's formalism for nonequilibrium statistical operator [3]. But these two procedures are rather approximate and not well justified.

Now it is becoming clear that for *nonlinear* systems a natural scenario of transition from reversible systems to irreversible ones should use the dynamical chaos conceptions [4]. For linear systems the well-justified correct procedure for creation of a *kinetic scenario* from the initial reversible equations is *absent*. In our opinion, this procedure for wide class of dynamical systems should include a formalism of the fractional calculus [5]. As it is well-known, the non-integer fractional operators of differentiation and integration have the property of the *partial* irreversibility or, in other words, the property of the 'remnant' memory considered in [6]. So, the problem of correct deduction of the fractional integral from usual integer integration operation is appeared.

Recently much attention has been paid to existence of equations containing *real* fractional exponent [7-9]. Now it becomes evident that equations with fractional derivatives will play a crucial role in description of kinetic and transfer phenomena in mesoscale region. As it was already discussed in paper [10] the frontiers of science are rapidly shifting from the investigation of the basic bricks of matter to the elucidation of mesoscopic principles of its organization. Moving in this way we need a mathematical apparatus, which adequately corresponds to a true description of *kinetic properties* of a matter on mesoscale region. From our point of view this necessary mathematical instrument should lie in deep understating of the 'physics' of the fractional calculus. The first attempt to understand the result of averaging of a smooth function over the given fractal (Cantor) set has been undertaken in [7]. In the note [11] and later in paper [12] some doubts were raised to the reliability of the previously obtained result. The criticism expressed in these publications forced the author (RRN) to reconsider the former result, and the detailed study of this problem showed that the doubts had some grounds and were directly linked with the relatively delicate questions of averaging a smooth function over fractal sets, in particular, on Cantor set and its generalizations. But we cannot agree with final conclusion made in [12]: "no direct relation between fractional calculus and the fractals has been established yet".

In order to dissipate these doubts and realize mathematically correct averaging procedure over fractal sets it was necessary to carry out a special study. This investigation has been given in the book [8], where the correct averaging procedure was considered in detail. The further generalization for more general Cantor sets has been realized in papers of Prof. Fu-Yao Ren with co-authors [13-15]. Another approach leading to the fractional integral and related to coarse graining time averaging is considered in the recent book [16]. Independent analysis of above-cited papers could lead to a conclusion that the physical meaning of the fractional integral with *real* exponent has been understood. Temporal fractional integral can be interpreted as a conservation of part of states localized on a self-similar (fractal) object if the physical system considered has at least two parts of different states. One part is distributed *inside* a fractal set (the conserved part of states) and another part of states is located *outside* of the fractal set (the lost part of states). That’s why it is easy to understand the fractional integral of one-half order, when for its understanding any fractal object is not necessary. Half of states is lost automatically in diffusion process with semi-infinite boundary conditions [8]. From the geometrical point of view the temporal fractional integral is associated with Cantor set or its generalizations, occupying an intermediate position between the classical Euclidean point and continuous line. But the meaning of fractional integral with real fractional exponent is *not* complete in the light of recent papers [10], [17-20], where the correct understanding of different self-similar objects with *complex* fractal dimension is discussed. These interesting ideas forced the authors of this paper to reconsider their previous results obtained in [8] and gave a possibility to understand the geometrical/physical meaning of mathematical operator with *complex* fractional exponent. So the basic question, which we are going to consider and discuss in this paper, can be formulated as follows: *how to come to understanding of fractional integral with complex fractional exponent through the correct averaging procedure of a smooth function over the temporal fractal set?* We are going to show that details of averaging procedure developed in [8] will help us to find the answer formulated in the title of this paper.

The following content of this article obeys the next structure. In the Section 2 we present the basic details of the averaging procedure including some new generalizations, which are absolutely necessary for further understanding. In Section 3 following to the basic ideas of the scale-invariance objects with complex fractal dimension we justify the geometrical/physical meaning of imaginary part of the complex fractional exponent. In this section the results of numerical calculations are also given. They are important in understanding of the imaginary part of the complex fractional exponent. We found also geometrical structures leading to a ‘pure’ fractional integration containing *only* an imaginary part of the complex exponent. This investigation (considered in Section 4) helps to introduce passive two-pole elements with *complex* fractional exponent, realizing the fractional integration/differentiation operation in time domain. The last Section 4 includes also a brief consideration of kinetic equations containing complex fractional exponent. The basic results are collected and discussed in the final Section 5.

2. Procedure of averaging of a smooth function over the given fractal set

2.1. Binary Cantor set and the temporal fractional integral

Let us suppose that a physical value $J(t)$ is related with a smooth function $f(t)$ by means of convolution operation

$$J(t) = K(t) * f(t) = \int_0^t K(t-\tau) f(\tau) d\tau \quad , \tag{1}$$

where the function $K(t)$ is determined on the segment $[0,T]$ and can be expressed by using of the conventional step-function

$$K(t) = \frac{1}{T}\left[\kappa(t) - \kappa(t-T)\right]. \tag{2}$$

Here

$$\kappa(t) = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases} \tag{3}$$

is the conventional Heaviside unit function. The constant $1/T$ in (2) appears as the result of the normalization of all states covered by the function $K(t)$ to the unit value

$$\int_0^T K(t) dt = 1. \tag{4}$$

Laplace-image of $K(t)$ with the use of retardation theorem

$$f(t-a) \overset{LT}{=} \exp(-pa) f(p), \tag{5}$$

takes the form

$$K(t) \overset{LT}{=} K(p) \equiv \int_0^\infty K(t) \exp(-pt) dt = \frac{1-\exp(-pT)}{pT}. \tag{6}$$

Here and below the symbol $\overset{LT}{=}$ means that the left and right side functions are related to each other by the conventional Laplace transform.

In order to find the kernel $K_{T,\nu}^{(N)}(t)$ on the Nth stage of the Cantor binary set construction, having fractal dimension $\nu = \ln 2/\ln(1/\xi)$ and concentrated on the interval $[0,T]$, it is necessary to write the recurrence relation directly for the kernel $K_{T,\nu}^{(N)}(t)$ which coincides with the normalized density of the binary set

$$K_{T,\nu}^{(N)}(t) = \frac{1}{2}\left[K_{\xi T,\nu}^{(N-1)}(t) + K_{\xi T,\nu}^{(N-1)}(t-(1-\xi)T)\right]. \tag{7}$$

Here $K_{T,\nu}^{(0)}(t) \equiv K(t)$ defined by (2). The height of each Cantor 'stripe' on the Nth stage is equaled to $1/(2\xi)^N T$ and provides the conservation of normalization to the unit on each stage of its construction.

In recurrence relation (7) and below the parameter ξ is the scaling factor, which shows the 'degree of compressing' ($\xi < 1$) of binary set on each stage of its construction. The values of ξ lie in the interval $[0,1/2]$. Now we are ready to find the answer for the following concrete question:

What is the result of the convolution of the function f(t) with the normalized density $K_{T,\nu}^{(N)}(t)$ in the limit $N \to \infty$, i.e

$$J(t) = \lim_{N\to\infty} J_N(t) = \lim_{N\to\infty}\left(K_{T,\nu}^{(N)}(t) * f(t)\right) = \left(K_{T,\nu}(t) * f(t)\right)? \tag{8}$$

Here and below $K_{T,\nu}(t)$ is the limiting value of $K_{T,\nu}^{(N)}(t)$. For further investigations of the last expression (8) it is convenient to use the Laplace transform of the function $K_{T,\nu}^{(N)}(t)$. From recurrence relationship (7) we obtain

$$K_{T,\nu}^{(N)}(p) = \frac{1}{2}[1+\exp(-pT(1-\xi))] K_{\xi T,\nu}^{(N-1)}(p). \tag{9}$$

Repeating this procedure N time, we have

$$K_{T,\nu}^{(N)}(t) \overset{LT}{=} K_{T,\nu}^{(N)}(p) = \frac{1-\exp\left(-pT\xi^n\right)}{pT\xi^n} Q_N\left[pT(1-\xi)\right], \tag{10}$$

where

$$Q_N(z) = 2^{-N}\prod_{n=0}^{N-1}(1+\exp(-z\xi^n)), \tag{11}$$

with $z = pT(1-\xi)$.

So, the Laplace-image of $J_N(p)$ accepts the form

$$J(p) = K_{T,\nu}^{(N)}(p) f(p) \tag{12}$$

For the large values of N ($N \gg 1$) and $\mathrm{Re}(pT\xi^N) \ll 1$ we have in the limit $N \to \infty$

$$J(p) = Q\left[pT(1-\xi)\right] f(p) = K_{T,\nu}(p) f(p) . \tag{13}$$

Here $Q[pT(1-\xi)]$ is the limiting value of the product (11). So, the limiting value of the integral kernel $K_{T,\nu}(p)$ is reduced to the investigation of the limiting value of the product (11).

2.2. Generalization for an arbitrary self-similar process

The last result (13) allows in generalizing of previous calculations for any self-similar process. One can notice that result (10) for $K_{T,\nu}^{(N)}(t)$ at $N \gg 1$ can be written in the form

$$K_{T,\nu}^{(N)}(t) = \Delta(t/T) * \Delta(t/\xi T) * ... * \Delta(t/\xi^{N-1}T) * K(t/\xi^N T) . \tag{14}$$

Here

$$\begin{gathered} \Delta(x) = \frac{1}{2}\left[\delta(x) + \delta(x-(1-\xi))\right] \\ K(t/\xi^N T) = \frac{1}{\xi^N T}\left[\kappa\left(t/\xi^N T\right) - \kappa\left((t/\xi^N T)-1\right)\right] \end{gathered}, \tag{15}$$

The analysis of formulae (14) and (15) prompts us to consider more general expressions for an arbitrary memory function $K^{(N)}(t)$ figuring in (14). Let us consider more general recurrencece relationship

$$K^{(N)}(t) = \beta_{N-1} g(\alpha_{N-1}t) * K^{(N-1)}(t) . \tag{16}$$

Here $g(t)$ is an arbitrary function, $\{\alpha_{i-1}\}$, $\{\beta_{i-1}\}$ (i=1,2,...,N,...) are sets of the constants. Applying the Laplace transform to the last expression we have

$$K^{(N)}(p) = \prod_{n=0}^{N-1} \frac{\beta_n}{\alpha_n} \hat{g}\left(\frac{p}{\alpha_n}\right). \tag{17}$$

We took into account the relationship

$$g(\alpha t) \overset{LT}{=} \frac{\hat{g}(\alpha p)}{\alpha} , \tag{18}$$

and initial condition

$$K^{(0)}(t) = \beta_0 g\left(\alpha_0 t\right) . \tag{19}$$

Relatively to product (17) we are making *two suppositions*:

S1. We put

$$\alpha_n = \beta_n = \frac{1}{T\xi^n} , \tag{20}$$

and write product (17) in the form

$$K^{(N)}(z) = \prod_{n=-(N-1)}^{N-1} \hat{g}(z\xi^n) = \prod_{n=0}^{N-1} \hat{g}(z\xi^n) \prod_{n=1}^{N-1} \hat{g}(z\xi^{-n}) . \tag{21}$$

Here we took into account the relationship (20) and extended the product also for *negative* values of n.

S2. We suppose that Laplace-image of $g(z)$ has the following decompositions

for $\mathrm{Re}(z) \ll 1$

$$\hat{g}(z) = 1 + c_1 z + c_2 z^2 + ... , \tag{22}$$

for $\mathrm{Re}(z) \gg 1$

$$\hat{g}(z) = \bar{g} + \frac{A_1}{z} + \frac{A_2}{z^2} + ... \tag{23}$$

Mathematical calculations realized in [8] show that:

(**A**) There is a limit of $K^{(N)}(z)$

$$\lim_{N\to\infty} K^{(N)}(z) \equiv K_\nu(z) = \frac{\pi_\nu(\ln(z))}{z^\nu} , \tag{24a}$$

where

$$\nu = \frac{\ln(\bar{g})}{\ln(\xi)}, \tag{24b}$$

and $\pi_\nu[\ln(z)]$ is a periodical function with the period of $\ln\xi$

$$\pi_\nu\left(\ln(z) \pm \ln\xi\right) = \pi_\nu\left(\ln(z)\right). \tag{25}$$

(**B**) The averaged value of the function $\pi_\nu[\ln(pT)]$ over the period $\ln\xi$ is defined by expression

$$C(\nu) \equiv \left\langle \pi_\nu(\ln(z)) \right\rangle = \int_{-1/2}^{1/2} \pi_\nu\left(\ln(z) + x\ln\xi\right) dx, \tag{26}$$

where the value of the constant $C(\nu)$ can be evaluated and equals

$$C(\nu) = \frac{1-\bar{g}}{\ln(1/\bar{g})} \exp\left[\frac{1}{\ln(\xi)} \int_0^\infty \ln(u) \frac{\hat{g}'(u)}{\hat{g}(u)} du\right]. \tag{27}$$

(**C**) In the limit $N \to \infty$ one can obtain

$$J(t) = \left\langle K_\nu(t) \right\rangle * f(t) = \frac{C(\nu)}{T^\nu} D^{-\nu}\left[f(t)\right] \equiv \frac{C(\nu)}{T^\nu \Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau \tag{28}$$

the desired relation with the fractional Riemann-Liouville integral

$$\left\langle K_\nu(t) \right\rangle \equiv \int_{-1/2}^{1/2} K_\nu(t\xi^{-x}) \left(\frac{\bar{g}}{\xi}\right)^x dx = \frac{C(\nu)}{T\Gamma(\nu)} \left(\frac{t}{T}\right)^{\nu-1}. \tag{29}$$

In the partial case

$$\hat{g}(z) = \frac{1+\exp(-z)}{2}, \tag{30}$$

for the binary Cantor set considered in the previous section the calculations give

$$\bar{g} = \frac{1}{2}, \quad C(\nu) = \frac{\exp\left(-\left(1+\frac{\nu}{2}\right)\ln 2\right)}{\ln(2)}. \tag{31}$$

The basic result of this section (24) can be easily understood if we notice the fulfillment of the following relationship, which is *exact* for any *finite N*

$$K^{(N)}(\xi z) = \frac{g(z\xi^N)}{g(z\xi^{-N+1})} K^{(N)}(z). \tag{32}$$

Taking into account the conditions (23) in the limit $N \to \infty$ relationship (32) for the fixed N is reduced to the scaling functional equation of the type

$$K(\xi z) = \frac{1}{\bar{g}} K(z), \tag{33}$$

having the solution (24) for any sequence $\{\xi: \xi_1\ \xi_2 \ldots \xi_N \ldots\}$ distributed on any *countable* set. If the sequence $\{\xi\}$ is *continuous* then we immediately restore the previous result [7] expressed by formulae (28) and (29).

3. The fractional integral with complex exponent

3.1. Consideration of the Cantor set with M bars. One mode approximation

The principal result (24) obtained in the previous section for wide class of functions $\hat{g}(z)$ with a variable z, which can accept real or complex values, helps to understand the meaning of fractional integral with the *complex* exponent. Following to ideas developed in [16] the periodical function with unit period can be expanded into the infinite Fourier series

$$\pi_\nu\left(\frac{\ln(z)}{\ln(\xi)}\right) = \sum_{n=-\infty}^{\infty} C_n \exp\left(2\pi n i \frac{\ln(z)}{\ln(\xi)}\right). \tag{34}$$

Taking into account the definition (24b) for ν and the last expression one can present expression (24) in the form

$$K_\nu(z) = \sum_{n=-\infty}^{\infty} C_n \exp\left[\left(\ln\left(\frac{1}{\bar{g}}\right) + 2n\pi i\right) \frac{\ln(z)}{\ln(\xi)}\right] = \sum_{n=-\infty}^{\infty} C_n \exp\left[\left(-\nu + i\Omega_n\right)\ln(z)\right]. \tag{35}$$

Here the real exponent ν is defined by expression (24b), $\Omega_n = 2\pi n/\ln\xi$ is a set of frequencies providing a periodicity with $\ln\xi$ of product (24a). Let us *suppose* that this infinite series can be replaced approximately by three terms

$$K_\nu(z) \cong z^{-\nu}\left(C_0 + A_n \exp(i<\Omega>\ln z) + A_n^* \exp(-i<\Omega>\ln z)\right)$$
$$= z^{-\nu}\left(C_0 + |A_n|\cos(<\Omega>\ln z - \psi)\right) = C_0 z^{-\nu} + A_n z^{-\nu+i<\Omega>} + A_n^* z^{-\nu-i<\Omega>} . \tag{36}$$

Here $<\Omega>$ is the averaged frequency referring to the leading term with an averaged value of n and the complex amplitude $A_n = |A_n|\exp(i\psi)$. These parameters determine the contribution of the leading term in the corresponding series (35). The value $<\Omega>$ is defined as

$$<\Omega> = \frac{2\pi <n>}{\ln \xi} . \tag{37}$$

For verification of expression (36) one can use the eigen-coordinates (ECs) method and consider the values C_0, $|A_n|$, $<\Omega>$ and ψ as a set of the fitting parameters. As an initial product one can take Laplace expression for M Cantor bars obtained in [8]

$$g_M(z) = \frac{1}{M}\frac{1-\exp\left(-\frac{zM}{M-1}\right)}{1-\exp\left(-\frac{z}{M-1}\right)} . \tag{38}$$

In particular case $M = 2$ this generalized expression coincides with (30). The basic principles of the ECs method have been considered in papers [21-25]. So, it is not necessary to repeat here the basic ideas. Here we are giving only the ECs for the function

$$y(z) = C_0 + |A_n|\cos(<\Omega>\ln(z) - \psi),\ K_\nu(z) \equiv z^{-\nu} y(z) . \tag{39}$$

In accordance with the ideology of the ECs method expression (39) initially including some nonlinear fitting parameters ($\langle\Omega\rangle, \psi$) can be transformed identically into the basic linear relationship

$$Y(x) = C_1 X_1(x) + C_2 X_2(x) + C_3 X_3(x) . \tag{40}$$

Here

$$\begin{aligned} &Y(x) = y - <\ldots> \\ &X_1(x) = \int_{x_0}^{x}(x-u)\,y(u)du - <\ldots>,\ C_1 = -<\Omega>^2, \\ &X_2(x) = x^2 - <\ldots>,\quad C_2 = \frac{<\Omega>^2 C_0}{2}, \\ &X_3(x) = x - <\ldots>,\quad C_3 = \left(<\Omega>^2 C_0 x_0 + y'(x_0)\right) \end{aligned} \tag{41}$$

is a linear combination of some functions depending on the variable $x = \ln(z)$. Relationship (40) helps to find two important fitting parameters C_0 and $<\Omega>$. Other two parameters $|A_n|$ and ψ are found from another basic linear relationship

$$U(x) = A_1 \cos(<\Omega> x) + A_2 \sin(<\Omega> x) , \tag{42}$$

where

$$U(x) = y(x) - C_0,\ A_1 = |A_n|\cos\psi,\ A_2 = |A_n|\sin\psi . \tag{43}$$

So, with the help of last expressions one can verify numerically the supposition (36) and calculate the necessary values of the fitting parameters C_0, $<\Omega>$, $|A_n|$ and ψ.

Numerical calculations are realized by means of the following procedure:

1. Calculation of the function $y(x)$ in accordance with definition (39).
2. Calculation of the fitting constants C_k (k=1,2,3) in accordance with linear relationship (40) by the *linear least-square method* (LLSM). They should present a set of sloping lines if supposition (36) is correct. The sloping lines C_1 and C_2 for $M = 2$ are shown in Fig.1
3. Calculation the necessary set of the fitting parameters (C_0, $<\Omega>$, $|A_n|$, ψ). The final verification of expressions (37). This final stage is presented by Figs. 2 and 3 respectively.

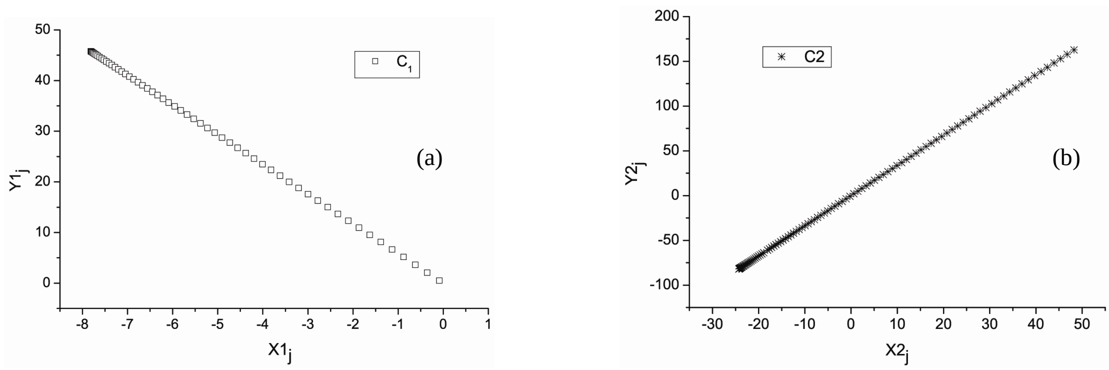


Fig.1. **(a)** *Plot for the constant C_1 calculated for number of bars M = 2. The sloping line indicates that the corresponding hypothesis for y(x) presented by (37) is correct. The tangent of the sloping line equals –5.85979.* **(b)** *Plot for the constant C_2 calculated for M = 2. Again the sloping line indicates that the corresponding hypothesis for y(x) presented by expression (37) is correct. The tangent of the sloping line equals 1.88859.*

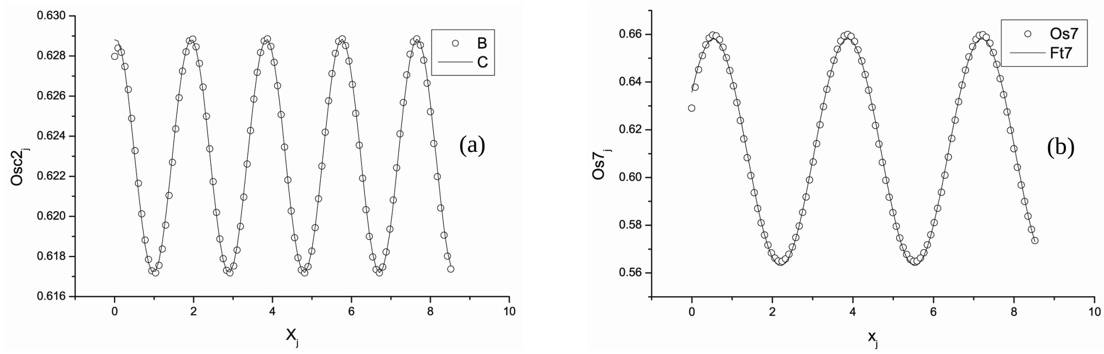


Fig.2. (***a***) *Oscillating part of y(x) shown by open points and its fit (shown by solid lines) calculated with the help of the ECs method for M = 2. The values of the fitting parameters are collected in Table I.* (***b***) *Oscillating part of y(x) shown by open points and its fit (shown by solid lines) calculated by the ECs method for M = 7. The values of the fitting parameters are collected in Table I.*

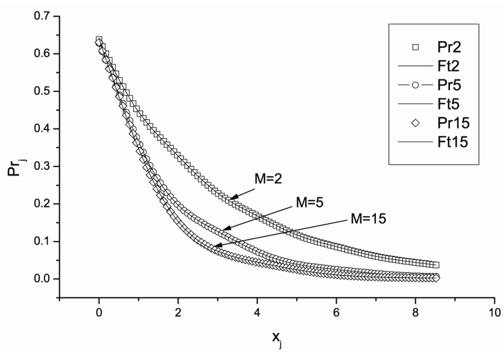


Fig.3. *Calculated values of the product corresponding to the function defined by expression (36) for different values of bars, which are defined by parameter M. Their fitting curves corresponding to function (37) are shown by solid lines. The values of the fitting parameters are collected in Table I.*

The values of the fitting parameters for various M and ξ are collected in Table I. These numerical calculations prove that supposition (36) is correct and physically reflects the true *discrete* structure of the fractal considered. It is interesting to note from analysis of the parameters given in Table I that the basic contribution to approximate expression (36) comes from the first Fourier components ($\langle n\rangle \cong 1$). Other parameters exhibit a monotonic behavior with respect to number of bars M. See, for example, Figs. 4a and 4b.

Table I. *The basic initial (the first 3 rows) and the fitting parameters (rest rows) obtained in the result of numerical verification of expressions (37)*

M	ξ	ν	C_0	A_0	A_1	Ω	ψ	<n>	Stdev
2	0.125	0.3333	0.63	0.0082	-0.0015	3.01161	-0.1748	0.9967	9.5E-5
5	0.05	0.5372	0.6117	0.0217	0.028	2.09144	0.9068	0.9972	6.1E-4
7	0.0357	0.584	0.6106	0.0252	0.0404	1.8853	1.0137	1.0000	8.3E-4
10	0.025	0.624	0.609	0.0293	0.0533	1.7045	1.06874	1.0001	1.3E-3
13	0.0192	0.6492	0.6074	0.0382	0.0621	1.5911	1.08429	1.1011	1.7E-3
15	0.0167	0.6614	0.606	0.0353	0.0661	1.5331	1.08074	0.9991	2.1E-3

Based on Laplace image (39) it is easy to come back in time-domain generalizing the value of the fractional Riemann-Liouville integral for complex values of the fractional exponent. Using relationship [26]

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \overset{LT}{=} p^{-\alpha}, \tag{44}$$

one can present expression (36) in the form

$$K_{\nu}(t) = C_0 \frac{t^{\nu-1}}{\Gamma(\nu)} + A_n \frac{t^{\nu+i<\Omega>-1}}{\Gamma(\nu+i\Omega)} + A_n^* \frac{t^{\nu-i<\Omega>-1}}{\Gamma(\nu-i\Omega)}. \tag{45}$$

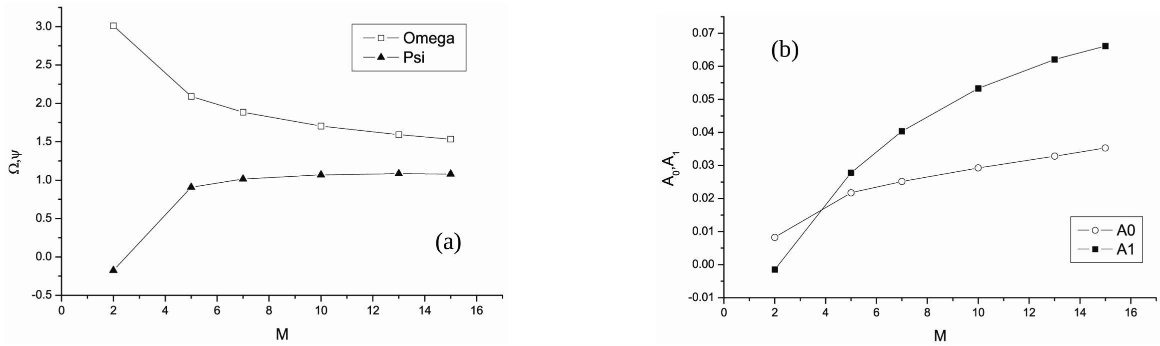


Fig.4. **(*a*)** *Dependence of the basic fitting parameters against the number of bars (M) is monotonic. Here we show the functions Ω(M) and ψ(M).* **(*b*)** *Dependence of the basic fitting parameters $A_0 = |A|\cos(\psi)$ and $A_1 = |A|\sin(\psi)$ against the number of bars (M) exhibits again a monotonic dependence. Other parameters are collected in Table I.*

It is easy to note that the first term in the last expression represents itself the evaluation of the kernel $K_\nu(t)$ in the *continual* approximation, other two terms reflect the discrete scale invariance phenomenon existing for true discrete fractals. The averaging procedure (see expression (26)) developed in the book [2] leads to zero values for the two last terms and effect of a "fractal digitization" is disappearing. So, coming back to objections expressed in papers of R. Rutman [11,12] one can say that initially this effect was not noticed and a "naive" attempt to replace a discrete product (17) by its continuous analog can be considered as *approximate*. The correct replacement requires the additional averaging procedure (26) or consideration of the fractal periodical effect, which in the simplest form can be expressed by two additional terms figuring in expression (45).

So, one can prove that the difference between random fractal, which accept any value of a scale from the given interval $(0,T)$ and discrete fractal that accept only *countable* set of scales leads to phenomenon of discrete scale invariance [4,17-20]. This phenomenon is expressed in the form of log-periodical functions with period depending on the scaling parameter $\ln\xi$. The eigen-coordinates method helps to identify the function $y(z)$ (39) and find the necessary fitting parameters C_0, $<\Omega>$, $|A_n|$ and ψ.

Attentive analysis of exact relationship (32) helps to find self-similar structures leading in time-domain to the complex fractional integral. Let us consider the additive sums of the following type appearing in averaging of a physical value over a discrete fractal structure [8]

$$S_N(z) = \sum_{n=-N+1}^{N-1} b^n f(z\xi^n). \tag{46}$$

Here and below the variable z can accept any real or complex value. This sum for any finite N has the following scaling property

$$S_N(z\xi) = \frac{1}{b} S_N(z) + b^{N-1} f(z\xi^N) - b^{-N} f(z\xi^{-N+1}). \tag{47}$$

If the function $f(z)$ is chosen in a way that contribution of the last two terms in the limit $N \to \infty$ becomes negligible then we obtain again the scaling equation of the type (33) with solution

$$S(\xi z) = \frac{1}{b} S(z), \quad S(z) = \frac{\pi(\ln(z))}{z^{\nu}}, \tag{48}$$

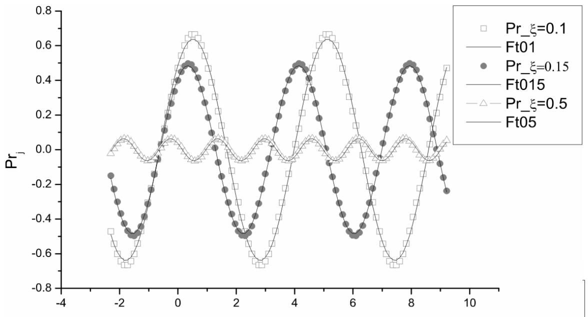
where $\pi(\ln(z))$ again is a log-periodical function, satisfying to condition (25), $\nu = \ln(b)/\ln(\xi)$. Solutions of the scaling equations of more general type obtained by variation of arbitrary constants are considered in the Mathematical Appendix. For $\overline{g}, b = \pm 1$ in Eqns. (33), (48) $\nu = 0$ and one can expect 'pure' log-periodical oscillations. For verification of these suppositions we chose two functions.

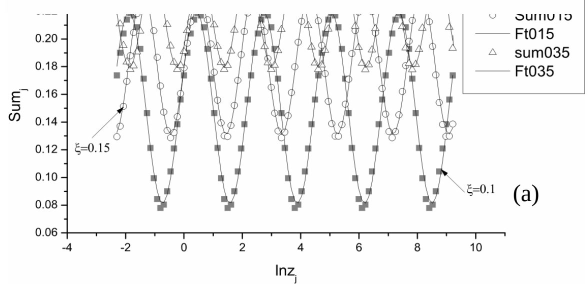
For the product (32) the probe function has a form

$$g(z\xi^n) = 1 - 2\cos(z\xi^n)\exp(-z\xi^n), \tag{49a}$$

which for $0 < \xi < 0.5$ provides the boundary conditions $g(z\xi^N) \cong -1$, $g(z\xi^{N+1}) \cong 1$.

For the sum (46) with $b = 1$ the function has the form





(a)

Fig.5. ***(a)*** *Numerical verification of the function (49a). Here we reproduce the product (32) calculated for ξ = 0.1, 0.15 and 0.5. The parameters of the fitting function defined by expression (36) are given in Table II. The range of variable z is located in the interval [0.1 ÷10000]. The function (25) at $\nu = 0$ satisfies to condition: $\pi_0\left(\ln(z) \pm \ln\xi\right) = -\pi_0\left(\ln(z)\right)$.* ***(b)*** *Numerical verification of the function (49b). Here we reproduce the calculated values of sum (46) for ξ = 0.1, 0.15 and 0.35. The parameters of the function defined by (36) are given in Table II. The range of variable z is located in the same interval [0.1 ÷10000]*

$$f(z\xi^n) = \left[1 - \cos(z\xi^n)\right]\exp(-z\xi^n)\ , \qquad (49b)$$

which for the same interval of the scaling parameter $0 < \xi < 0.5$ provides zero boundary conditions $f(z\xi^{-N}) \cong 0$, $f(z\xi^{-N+1}) \cong 0$. The set of Figs. 5 depict the desired oscillations obtained for 'pure' complex' case with $\nu = 0$. The parameters of the fitting function (39) for some values of ξ are given in Table II. Finishing this section one can say that possible complex solution of the scaling equation (33) always exist for one of the negative values $g(z\xi^N)$, $g(z\xi^{-N+1})$. For this case we have the solution

$$K(z) = \frac{\pi(\ln(z))}{z^{\nu}},\ \pi(\ln(z) \pm \ln(\xi)) = -\pi(\ln(z)) \qquad (50)$$

with $\nu = \ln(1/\bar{g})/\ln(1/\xi)$ **.** So, decomposition of the log-periodic function for this case into the Fourier series should contain only odd components

(b)

$$S_N(z) = \sum_n \left[\varphi(n)\right]^{\beta} F(z\left[\varphi(n)\right]^{\alpha})\ . \qquad (51)$$

Table II. *The calculated fitting parameters obtained for product (32) with function (49a) (the first nine rows marked bold) and sum (46) with function (49b) (the last eight rows).*

ξ	C_0	A_0	A_1	Ω	ψ	<n>	Stdev
0.1	**-0.24E-4**	**0.486081**	**0.414370**	**1.361998**	**0.705923**	**0.499128**	**0.00964**
0.15	**3.927E-5**	**0.410544**	**0.258893**	**1.653635**	**0.5626226**	**0.499293**	**0.005364**
0.2	**-1.3E-5**	**0.33813**	**0.13424**	**1.94831**	**0.377924**	**0.499059**	**0.0038366**
0.25	**6.88E-5**	**0.27102**	**0.046309**	**2.26341**	**0.1692346**	**0.499389**	**0.003052**
0.3	**2.163E-6**	**-0.203831**	**0.017272**	**2.599207**	**0.08453686**	**0.49805546**	**0.003325**
0.35	**-7.62E-7**	**-0.1404406**	**0.0481402**	**2.97784**	**0.330228**	**0.4975498**	**0.0031357**
0.4	**1.948E-8**	**0.0820374**	**-0.057847**	**3.406445**	**0.6141585**	**0.4967693**	**0.0031405**
0.45	**-1.63E-7**	**0.03623903**	**-0.0530627**	**3.901106**	**0.971607**	**0.495778**	**0.0033237**
0.5	**2.288E-8**	**-0.006377**	**0.0374794**	**4.481137**	**1.402258**	**0.4943492**	**0.0029298**
0.1	0.150513	0.0284258	0.0630613	2.7203411	1.14731	0.996917	0.0023746
0.15	0.1826834	-0.0031946	0.051340	3.29363	1.50865	0.994466	0.0015673
0.2	0.21534	-0.020898	0.0292423	3.871489	-0.9503057	0.991682	0.001787
0.25	0.25	-0.0227797	0.0068779	4.4808124	-0.2932225	0.9886267	0.0017665
0.3	0.287858	-0.0130835	-0.0070482	5.1412523	0.4941338	0.985158	0.0017137
0.35	0.33012601	-0.0011869	-0.0082609	5.867455	1.428099	0.9803601	0.0014721
0.4	0.378235	0.0033914	-0.002246	6.6804012	0.585039	0.9742176	0.001067
0.45	0.434027	0.0010721	0.0010717	7.6022915	0.785196	0.9661482	0.0006704

3.2 Some generalizations

Let us consider the sum or the product of the following type

$$S_N(z) = \sum_n \left[\varphi(n)\right]^{\beta} F(z\left[\varphi(n)\right]^{\alpha})\ , \qquad (52a)$$

$$K^{(N)}(z) = \prod_n g\left(z\varphi(n)^{\alpha}\right). \qquad (52b)$$

It is supposed that the values of the discrete variable *n* is located in the interval, which keeps the real values of the function $\varphi(n)$. These expressions can be transformed to the following forms

$$S_N(z) = \sum_n \left[\varphi(n)\right]^{\beta} F(z\left[\varphi(n)\right]^{\alpha}) = \sum_{K=-(N-1)}^{N-1} b^K F(z\xi^K)\ , \qquad (53a)$$

$$K^{(N)}(z) = \prod_{n} g\left(z\varphi(n)^{\alpha}\right) = \prod_{K=-(N-1)}^{N-1} g(z\xi^{K}) . \tag{53b}$$

Here $b = \exp(\beta)$, $\xi = \exp(\alpha)$, $K = \ln[\varphi(n)]$. If the initial values of n are chosen in such a way that discrete values of K are located in the interval $[-N + 1 \le K \le N - 1]$ then the mapping $\varphi(n) = \exp(K)$ keeps *invariant* all properties proved initially for product (32) and finite sum (46). So, one can say that fractional integral with real or complex exponent exists not only for genuine fractal structures. The mapping $\varphi(n) = \exp(K)$ considerably increases the results obtained for new type of structures, which can be reduced to the fractal ones on mesoscale region. It has a sense to define these structures as *quasi-fractals* structures. In particular, in chapter 8 of the book [8] we consider the model of coordination spheres, when $n \equiv \varphi(n)$. As numerical verifications show the dependences $N_n = N_0 n^{\alpha}$ and $R_n = R_0\, n^{\beta}$ approximate very well the number of particles N_n and their radiuses R_n as a function of a number of the current coordination sphere n (n = 0,1,2,…). The model of coordination spheres can be applied for calculation of number of particles for wide number of heterogeneous substances including clusters of different nature. No needless to say that similar quasi-fractal structures leading also to the fractional integral of the Riemann-Liouville type are needed in more detailed investigations as new potential objects figuring in mesoscale region.

4. Recap and reind elements with complex exponents

4.1. Possibility of existence of reind and recap elements with complex exponents

In our book [8] it has been proved that self-similar structures combined from R (resistance), C (capacitance) and L (inductance) elements form passive two-poles, which we defined as *recap* (*re*sistance + *cap*acitance) and *reind* (*re*sistance + *ind*uctance) elements. Their impedances are expressed in the form

$$\begin{aligned} Z_{\nu}(j\omega) &= R\left(j\omega\right)^{-\nu}\ (0 \le \nu \le 1) \\ Z_{\nu}(j\omega) &= R\left(j\omega\right)^{\nu}\ (0 \le \nu \le 1) \end{aligned} . \tag{54}$$

The first expression defines the complex impedance of recap; the second one belongs to reind element. All analytical evaluations, which lead to expressions (54) were performed in the continual limit [8]. One can expect that calculations realized for true discrete structures will contain log-periodic functions, reflecting discrete scale invariance phenomenon. In fact, these log-periodic functions are contained definitely as solutions of the scaling equation

$$\Phi(z\xi) = b\Phi(z) , \tag{55}$$

if parameters b and ξ form independent countable sets [8]. For other fractal structures, which are not satisfied to functional equation (55) the existence of log-periodic solutions needs in a special investigation. Consider, for example, the chain of two elements: resistance R and capacitance $Z = R/(j\omega\tau\xi^{n})$ ($\tau = RC$, $\xi < 1$) connected *in series*. The total admittance of these elements ($-N + 1 < n < N - 1$) connected *in parallel* is expressed in the form [2]

$$Y_N(z) = \frac{1}{R}\left(1 - S_N(z)\right). \tag{56a}$$

If these two elements R and $Z = R/(j\omega\tau\xi^{n})$ ($\tau = RC$, $z = j\omega\tau$) are connected *in parallel*, then the total impedance of these elements connected *in series* is expressed in the form

$$Z_N(z) = RS_N(z) . \tag{56b}$$

For ‘extraction’ of log-periodic solutions we consider the sum

$$S_N(z) = \sum_{n=-(N-1)}^{N-1} \frac{1}{1 + z\xi^{n}} , \tag{57}$$

figuring in both expressions (56). According to expression (47) we have the following scaling equation (if $f(z\xi^{N}) \cong 1$, $f(z\xi^{N+1}) \cong 0$)

$$S(z\xi) = S(z) + 1 . \tag{58}$$

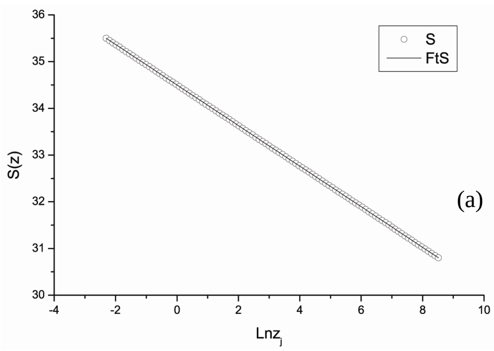
Solution of this equation (see Mathematical Appendix) can be written in the form

$$S(z) = \pi\left(\ln(z)\right) + \frac{\ln(z)}{\ln\xi} . \tag{59}$$

Figs. 6a and 6b show the results of numerical verification of solution (59). In Fig. 6a we depict the situation, when possible oscillations are completely *hidden* and suppressed totally by the second term $\ln(z)/\ln(\xi)$. After subtraction of the second term in (59) possible oscillations evoked by discrete circuit structure become visible. Oscillations $S(z) - \ln(z)/\ln(\xi)$ shown on Fig. 6b are described well by the fitting function

$$\begin{aligned} \pi\left(\ln(z)\right) &= C_0 + C_1 \exp(i < \Omega > \ln(z)) + C_1^{*} \exp(-i < \Omega > \ln(z)) \\ < \Omega > &= \frac{2\pi < n >}{\ln(1/\xi)} \end{aligned} , \tag{60}$$

with the following values of the fitting parameters $\xi = 0.1$, $C_0 = 34.5$, $C_1 = A_1\exp(i\psi)$ (A_1 = -3.19382E-5, ψ = 1.53985, $\langle n\rangle = 0.996368$, $\langle\Omega\rangle = 2.71884$). The standard deviation of the absolute difference between the left and the right parts of expression (60) equals Stdev = 1.45291E-5.



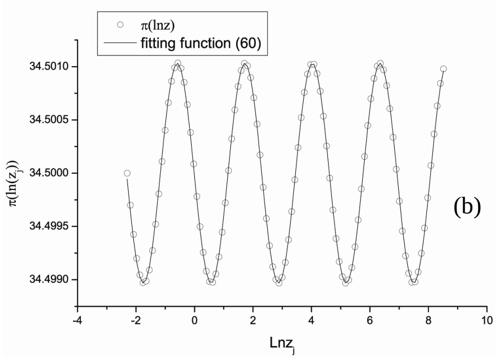


Fig.6. **(*a*)** *The calculated sum defined by expression (59) (gray points) and the fitting function (solid line) defined by expression y(z) = ln(z)/ln(ξ) + B (ξ = 0.1, B = 34.5115). In this presentation a possible discrete structure is completely hidden. In order to see possible oscillations it is necessary to analyze the difference expressed by (60).* **(*b*)** *The log-periodic function and its fitting (expression (60)) obtained by the ECs method. The fitting parameters calculated for ξ = 0.1 are given in the text.*

In more general cases for *discrete* self-similar structures one can write the following generalization of expressions (54)

$$Z_\nu(j\omega) = R\left[(j\omega)^{-\nu} + C(j\omega)^{-\nu+j\Omega} + C^*(j\omega)^{-\nu-j\Omega}\right] \; (0 \le \nu \le 1)$$
$$Z_\nu(j\omega) = R\left[(j\omega)^{\nu} + C(j\omega)^{\nu+j\Omega} + C^*(j\omega)^{\nu-j\Omega}\right] \; (0 \le \nu \le 1) \tag{61}$$

Here R is a dimension value, C is a dimensionless complex constant of the order of unity, Ω is a leading frequency defined by expression (60). The structure of expressions (61), when terms containing complex exponents form a *complex-conjugated* pair follows from general expression (34).

The existence of the generalized expressions (61) is confirmed by numerical calculations but it would be interesting to discover this behavior in real experimental situations analyzing impedances/admittances frequency behavior of various heterogeneous structures.

4.2. 'Strange' fractal kinetics

A strong evidence has been presented earlier [9,27,28] that the generalized kinetic equations containing fractional derivatives and integrals describe well raw dielectric spectroscopy (DS) data, which, in turn, are related to measurements of complex permittivity in frequency domain. For this aim the special recognition procedure has been developed. It includes a presentation of DS data in the so-called *ratio* format. The special *separation* procedure helps to identify the number of relaxation processes and determines at least qualitatively the possible structure of fractional equation describing the relaxation of the total polarization in time-domain. Experimental DS data are described very well by complex permittivity functions, which correspond to new identified kinetic equations with fractal derivatives in time-domain. As a *basic* result, which follows from this new approach one can obtain a new interpretation of the empirical Vogel-Fulcher-Tamman (VFT) equation together with its possible corrections [27]. This equation taken in the conventional and generalized forms describes the temperature dependence of low-frequency loss peak for wide class of heterogeneous materials. These papers can be considered as an essential argument that "fractal" kinetics really exists in nature. These identified and recognized kinetic equations have the following forms

$$\left(\tau_1^{\nu_1} D_{t_0}^{\nu_1} + \tau_2^{\nu_2} D_{t_0}^{\nu_2}\right)\left(P(t) - P(t_0)\right) + P(t) = 0,$$
$$\left(\tau_1^{-\nu_1} D_{t_0}^{-\nu_1} + \tau_2^{-\nu_2} D_{t_0}^{-\nu_2}\right)^{-1}\left(P(t) - P(t_0)\right) + P(t) = 0. \tag{62}$$

Here $P(t)$ is a value of the total polarization, $\tau_{1,2}$ are characteristic relaxation times, $\nu_{1,2}$ are fractional exponents located presumably in the interval [0,1]. It is interesting to mark that the second equation in (47) contains a linear combination of fractional *integral* operations, but this combination taken in inverse degree gives again a *specific fractional derivative*. The stationary solutions of these equations lead to the following expressions for complex susceptibility.

$$\varepsilon(j\omega) = \varepsilon_\infty + \frac{\varepsilon(0) - \varepsilon_\infty}{1 + (j\omega\tau_1)^{\nu_1} + (j\omega\tau_2)^{\nu_2}},$$
$$\varepsilon(j\omega) = \varepsilon_\infty + \frac{\varepsilon(0) - \varepsilon_\infty}{1 + \left[(j\omega\tau_1)^{-\nu_1} + (j\omega\tau_2)^{-\nu_2}\right]^{-1}}. \tag{63}$$

These expressions help to give a new interpretation of the VFT equation. This statement is confirmed by independent verification of randomly taken (from various international dielectric laboratories) raw DS data measured for complex permittivity.

These identified kinetic equations can be easily generalized. More general form of kinetic equation which can be considered as a potential candidate for description of DS complex permittivity data in wide class of heterogeneous materials can be written in the form

$$\sum_{k=1}^{n}\left(\tau_k^{\nu_k} D_{t_0}^{\nu_k}\right)\left(P(t)-P(t_0)\right)+P(t)=0\cdot \tag{64}$$

In partial cases ($n=1$, $\nu=1$ and $n=1$, $\nu\neq 1$) the last kinetic equation for the total polarization $P(t)$ coincides correspondingly with the known kinetics of the Debye's and Cole-Cole's type. The physical meaning of the last kinetic equation is the following. We suppose that all relaxation system including a set of strongly correlated microdipoles can be divided on n subsystems. It might be a set of dipole clusters or ensemble of strongly correlated molecules. Each subsystem is interacting with thermostat with the help of collision/rotation mechanism, which is expressed by means of fractional derivative. Each subsystem k (k = 1,2,…,n) is characterized by own characteristic relaxation time τ_k showing the contribution of the chosen relaxation unit into the general process of relaxation. The number of subsystems, giving an additive contribution to the general picture of relaxation, is defined by a structure of the concrete heterogeneous material considered. At an initial stage the kinetic equation (64) can be considered as a reasonable and *phenomenological* hypothesis, which is recognized from correct treatment of DS data. After identification of this type of kinetic equation on a wide class of heterogeneous materials the further theoretical attempts should be undertaken in explanation of their *microscopic* origin. Probably, it will require the generalization of the Liouville equation for density matrix and introduction of new ideas related to irreversibility of time. At the present stage we suppose that this equation describing the relaxation of the total polarization in a bulk material can serve a basis of signal processing in the modern dielectric spectroscopy.

Now it becomes clear *how* to generalize the identified kinetic equation of the type (62) for the *complex* fractional exponents and clarify their physical meaning. Let us *suppose* that discrete scale structure in the heterogeneous material considered is conserved. The reasons of conservation of a discrete structure in some concrete material need a special consideration. If some materials exhibit the discrete scale invariant (DSI) property then it is necessary to replace a real fractional exponent by the triad of the following type

$$\tau_k^{\nu_k} D_{t_0}^{\nu_k} \Rightarrow \tau_k^{\nu_k} D_{t_0}^{\nu_k} + C\tau_k^{\nu_k+j\Omega_k} D_{t_0}^{\nu_k+j\Omega_k} + C^{*}\tau_k^{\nu_k-j\Omega_k} D_{t_0}^{\nu_k-j\Omega_k}\ . \tag{65}$$

The last two terms in (65) reflect the influence of a possible DSI property of a self-similar structure into the general process of relaxation. *Physically* this replacement can be interpreted as relaxation process taking place on a discrete log-periodical structure with a basic mode $<\Omega>$ and having the statesν, which are kept on this structure. For random fractals the effect of log-periodicity is lost and only the real part of the total complex exponent is conserved. So, one can see the close relationship between geometrical (structural) and physical (relaxation) properties taking place on log-periodical self-similar structures. The most fascinating thing which follows from this generalization is the prediction of a ‘strange’ (unusual) kinetics, when the complex exponents accept a weak dependence on time. The contribution of one complex exponent coming from a genuine discrete structure into the general picture of relaxation can be presented in the form

$$\begin{aligned}&\left(\tau^{\nu(t)} D_{t_0}^{\nu(t)} + C\tau^{\nu(t)+j\Omega(t)} D_{t_0}^{\nu(t)+j\Omega(t)} + C^{*}\tau^{\nu(t)-j\Omega(t)} D_{t_0}^{\nu(t)-j\Omega(t)}\right)\left(P(t)-P(t_0)\right)+\\ &\qquad +P(t)=0\end{aligned}\ . \tag{66}$$

The kinetic equations of the type (64) and (66) with possible inclusion and evolution in time of complex fractional exponents (if their existence will be definitely proved in the nearest future) can require a deep reconsideration of the basics of the modern nonequilibrium statistical mechanics. We suppose that understanding of the physical meaning of complex fractional exponent opens new directions not only for dielectric spectroscopy, where the corresponding kinetic equations containing fractional integral/derivatives have been identified. It will give a stimulus for other branches of physics and chemistry of heterogeneous materials, where the discovery of fractional kinetics with any value of derivative (including the complex exponents) is still waiting its proper time.

5. Results and discussion

Based on the scale invariance property, which exists for fractals with clearly expressed discrete structure, it becomes possible to understand the geometrical/physical meaning of the fractional derivatives and integrals with complex fractional exponents. The true form of this complex structure, which can enter into kinetic equation with fractional derivative, has been found. These kinetic equations can solve the problem of the correct deduction of irreversibility phenomenon for *linear* systems. The fractional derivative with complex exponent should enter into a linear kinetic equation as a structure containing three basic terms. The first term reflects a possible continuous structure. Other two complex-conjugated terms reflect a log-periodicity of a scale, which forms the discrete fractal structure considered. This complex triad structure is confirmed by numerical calculations. We found also possible structures when real exponent $\nu = 0$. It can be happened at consideration of sum (46) at b = 1 and product (32) when one of the limiting value becomes *negative* or these values in asymptotic limit coincides with each other. We found also *quasi-fractal* objects, which keep invariant the Riemann-Liouville definition of the fractional integral with real and complex exponents. These objects considerably increase the applicability of the conception of the fractional integral obtained previously for true fractal structures. The understanding of geometrical meaning of complex fractional exponents helps to generalize the conception of recap and reind elements and put forward an idea of existence of kinetic equations with complex fractional derivatives. The conception of the fractional exponent helps to understand deeper the principal difference between discrete and continuous organization of a matter on mesoscale region and identify a set of genuine discrete self-similar structures, taking part in transfer and relaxation processes.

Mathematical Appendix

The solution of the generalized scaling equations

Here we want to generalize scaling equations of the type (32) and (47) and give their solutions obtained by the method of an arbitrary constant variation. Another method of finding of solutions of some set of scaling equations was considered in [29].

At first, one can notice that it is easier to obtain solutions of the generalized scaling equation (47). The corresponding solution of the generalized equation (32) is obtained by ordinary exponentiating. In the Table III given below we use the following notations: $\pi(\ln(z))$ is a log-periodic function, which in the most cases can be expressed in the form

$$\begin{gathered}\pi\left(\ln(z)\right) = C_0 + C_1 \exp(i < \Omega > \ln(z)) + C_1^* \exp(-i < \Omega > \ln(z)),\\ < \Omega > = \frac{2\pi < n >}{\ln(\xi)}\end{gathered} \qquad \text{(A1)}$$

The fitting parameters of this function can be found with the help of the eigen-coordinates method.

Table III.

Scaling Equation	Solution	Comments
1. $(b \neq 1)$ $S(z\xi) = bS(z) + c_0 + \sum_{k=1}^{n} c_k z^k + c_{-k} z^{-k}$ $K(z\xi) = A\left[K(z)\right]^b \exp\left[\sum_{k=1}^{n}\left(c_k z^k + c_{-k} z^{-k}\right)\right]$ $(b = 1)$ $S(z\xi) = S(z) + c_0 + \sum_{k=1}^{n} c_k z^k + c_{-k} z^{-k}$ $K(z\xi) = \left[K(z)\right]\exp\left[c_0 + \sum_{k=1}^{n}\left(c_k z^k + c_{-k} z^{-k}\right)\right]$	$S(z) = z^{\nu}\pi(\ln(z)) + \frac{c_0}{1-b} + \sum_{k=1}^{n}\left(\frac{c_k z^k}{\xi^k - b} + \frac{c_{-k} z^{-k}}{\xi^{-k} - b}\right)$ $K(z) = \exp\left(S(z)\right)$ $S(z) = \pi \ln(z)\frac{c_0}{\ln(\xi)}\ln(z) + \sum_{k=1}^{n}\left(\frac{c_k z^k}{\xi^k - 1} + \frac{c_{-k} z^{-k}}{\xi^{-k} - 1}\right)$ $K(z) = \exp\left(S(z)\right)$	The limits of applicability: $c_{n+1} z^{n+1} \ll 1$ $c_{-n-1} z^{-n-1} \ll 1$ $K(z)$ can be considered as scaling equation for the stretched exponential function.
2. $(b \neq 1)$ $S(z\xi) = bS(z) + a\ln(z) + c_0$ $K(z\xi) = e^{c_0} z^a \left[K(z)\right]^b$ $(b = 1)$ $S(z\xi) = S(z) + a\ln(z) + c_0$ $K(z\xi) = e^{c_0} z^a \left[K(z)\right]$	$S(z) = z^{\nu}\pi\left(\ln(z)\right) + \frac{a}{1-b}\ln(z) + \frac{c_0}{1-b} - \frac{a\ln(\xi)}{(1-b)^2}$ $S(z) = \pi(\ln(z)) + \frac{a}{2\ln(\xi)}(\ln z)^2 + \left(\frac{c_0}{\ln\xi} - \frac{a}{2}\right)\ln z$ $K(z) = \exp\left(S(z)\right)$	One can add the term $\sum_{k=1}^{n}\left(c_k z^k + c_{-k} z^{-k}\right)$ and obtain the corresponding solution.

The variable z accepts any value and can be real or complex, $\nu = \ln(b)/\ln(\xi)$.

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